

# Convergence of a Grassmannian Gradient Descent Algorithm for Subspace Estimation From Undersampled Data

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## Abstract

Subspace learning and matrix factorization problems have a great many applications in science and engineering, and efficient algorithms are critical as dataset sizes continue to grow. Many relevant problem formulations are non-convex, and in a variety of contexts it has been observed that solving the non-convex problem directly is not only efficient but reliably accurate. We discuss convergence theory for a particular method: first order incremental gradient descent constrained to the Grassmannian. The output of the algorithm is an orthonormal basis for a  $d$ -dimensional subspace spanned by an input streaming data matrix. We study two sampling cases: where each data vector of the streaming matrix is fully sampled, or where it is undersampled by a sampling matrix  $A_t \in \mathbb{R}^{m \times n}$  with  $m \ll n$ . We propose an adaptive stepsize scheme that depends only on the sampled data and algorithm outputs. We prove that with fully sampled data, the stepsize scheme maximizes the improvement of our convergence metric at each iteration, and this method converges from any random initialization to the true subspace, despite the non-convex formulation and orthogonality constraints. For the case of undersampled data, we establish monotonic improvement on the defined convergence metric for each iteration with high probability.

**Keywords:** Subspace learning, Non-convex formulation, Grassmannian Gradient Descent, Global Convergence, Adaptive stepsize

## 1. Introduction

The low-rank subspace model is an essential tool for high-dimensional inference with fewer measurements than variables of interest, where low-dimensional models are necessary to perform accurate and stable inference. Many modern problems fit this paradigm, where signals are undersampled because of sensor failure, resource constraints, or privacy concerns. Suppose we wish to factorize a matrix  $M = UW^T$  when we only get a small number of linear measurements of  $M$ . Solving for the subspace basis  $U$  can be computationally burdensome in this undersampled problem and related regularized problems, and many algorithms that attempt to speed up computation are actually solving a non-convex optimization problem, therefore coming with few guarantees.

The Singular Value Decomposition (SVD) is the solution to a non-convex optimization problem, and there are several highly successful algorithms for solving it [Golub and Van Loan \(2012\)](#). Unfortunately, these algorithms cannot easily be extended to problems with incomplete observations of the matrix of interest. Recently, several results have been published with first-of-their-kind

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guarantees for a variety of different gradient-type algorithms on non-convex matrix factorization problems Jain et al. (2013, 2016); De Sa et al. (2015); Armentano et al. (2014); Chen and Wainwright (2015); Bhojanapalli et al. (2015); Zheng and Lafferty (2015). These new algorithms, being gradient-based, are well-suited to extensions of the SVD where the matrix is not fully sampled and where we include different cost functions or regularizers. For example, with gradient methods to solve the SVD we may be able to solve Robust PCA Candès et al. (2011); He et al. (2012); Xu et al. (2010), Sparse PCA d’Aspremont et al. (2008), or even  $\ell_1$  PCA Brooks et al. (2013) with gradient methods as well.

Our contribution is to provide a global convergence result for  $d$ -dimensional subspace estimation using an incremental gradient algorithm performed on the Grassmannian, the space of all  $d$ -dimensional subspaces of  $\mathbb{R}^n$ , denoted by  $\mathcal{G}(n, d)$ . Subspace estimation is a special case of matrix factorization with orthogonality constraints, where we seek to estimate only the subspace spanned by the columns of the left matrix factor  $U \in \mathbb{R}^{n \times d}$ . Our result demonstrates that, for fully sampled data without noise, this gradient algorithm *converges globally* almost surely, *i.e.*, it converges from any random initialization to the global minimizer. To the best of our knowledge, Zhang and Balzano (2015) provided the first global convergence result for an incremental gradient descent method on the Grassmannian. Here we simplify the analysis of Zhang and Balzano (2015) and provide a slightly tighter bound on the number of iterations required to get to a local region of the global optimal point. We also extend these results to undersampled data, both compressive measurements and missing data. Together with the simplified full-data results, this paper provides a unified framework to show that in all cases, our algorithm provides a sequence of subspace estimates  $U_t$  whose expected improvement at each iteration is of the form

$$\mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} \middle| U_t \right] \geq 1 + \eta \frac{m}{n} \frac{1 - \zeta_t}{d}$$

where  $\zeta_t$  is a metric of subspace similarity and goes to 1 as the subspaces converge,  $\eta \approx 1$  is slightly different for each problem,  $n$  is the ambient dimension,  $d$  is the rank, and  $m$  the number of measurements. We provide monotonic improvements on the metric of convergence in terms of expectation for each iteration. Comparing with the fully sampled case, we prove that the reduction in expected improvement is proportional to the sampling density. We show empirically in a variety of simulations that this bound is precise.

## 2. Formulation and Related Work

### 2.1. Problem Setting

In this paper, we consider the problem of learning a low dimensional subspace representation from streaming data. Specifically, we are given a sequence of observations  $x_t = A_t v_t$  where  $A_t \in \mathbb{R}^{m \times n}$  ( $m \leq n$ ) are sampling matrices that are given for each observation; and  $v_t \in \mathbb{R}^n$  are drawn from a continuous distribution with support on the true subspace, spanned by  $\bar{U} \in \mathbb{R}^{n \times d}$  with orthonormal columns, *i.e.*,  $v_t = \bar{U} s_t$ ,  $s_t \in \mathbb{R}^d$ . In this paper, we study three different sampling frameworks: fully sampled case with  $A_t$  being the identity matrix, the compressively sampled case with  $A_t \in \mathbb{R}^{m \times n}$  ( $m \ll n$ ) being random matrices, and the missing data case where each row of  $A_t$  is uniformly sampled from the identity matrix.

We formulate subspace estimation as a non-convex optimization problem as follows. Let  $U \in \mathbb{R}^{n \times d}$  be a matrix with orthonormal columns. Then we want to solve:

$$\begin{aligned} & \underset{U \in \mathbb{R}^{n \times d}}{\text{minimize}} && \sum_{t=1}^T \min_{w_t} \|A_t U w_t - x_t\|^2 \\ & \text{subject to} && \text{span}(U) \in \mathcal{G}(n, d) \end{aligned} \quad (1)$$

This problem is non-convex firstly because of the product of the two variables  $U$  and  $w_t$  and secondly because the optimization is over the Grassmannian  $\mathcal{G}(n, d)$ , the non-convex set of all  $d$ -dimensional subspaces in  $\mathbb{R}^n$ . We study an online algorithm to solve the above problem, where we process one observation at a time and perform a rank-one update to generate a sequence of estimates  $U_t$  with the goal that  $R(U_t) \rightarrow R(\bar{U})$ , where  $R(\cdot)$  denotes the column range.

With a modification of (1), we can see the relationship between our problem and the well studied low-rank matrix recovery problem. Let  $W \in \mathbb{R}^{d \times T}$  and  $M = [v_1, \dots, v_T] \in \mathbb{R}^{n \times T}$ , then (1) is equivalent to

$$\begin{aligned} & \underset{U \in \mathbb{R}^{n \times d}, W \in \mathbb{R}^{d \times T}}{\text{minimize}} && \sum_{t=1}^T \min_{w_t} \|\mathcal{A}(UW) - \mathcal{A}(M)\|^2 \\ & \text{subject to} && \text{span}(U) \in \mathcal{G}(n, d) \end{aligned} \quad (2)$$

where  $\mathcal{A} : \mathbb{R}^{n \times T} \rightarrow \mathbb{R}^{m \times T}$  is a linear operator. Our algorithm can be thought of as an incremental algorithm to solve this problem as well. The formulation in (2) also allows for convex relaxations, but unfortunately (1) does not. Still, fueled by the great deal of recent success of directly solving non-convex factorization problems (as we discuss in related work below), we study the natural gradient descent algorithm applied to (1) directly. Since the optimization variable in our problem is a subspace, we constrain the gradient descent to the Grassmannian  $\mathcal{G}(n, d)$ . The resulting algorithm is called GROUSE (Grassmannian Rank-One Update Subspace Estimation) algorithm [Balzano et al. \(2010a\)](#), and is described in Algorithm 1.

## 2.2. Related Work

Many recent results have shown theoretical support for directly solving non-convex matrix factorization problems with gradient or alternating minimization methods. Among the incremental methods [De Sa et al. \(2015\)](#) is the one closest to ours, where the authors consider recovering a positive semidefinite matrix with undersampled data. They propose a step size scheme with which they prove global convergence results from a randomly generated initialization. However, their choice of step size depends on the knowledge of some parameters that are likely to be unknown in practical problems. Without this knowledge, the results only hold with sufficiently small step size that implies slower convergence. In contrast, our step size only depends on the observations and outputs of the algorithms. Similarly, [Balsubramani et al. \(2013\)](#) invokes a martingale-based argument to show the global convergence rate of the proposed incremental PCA method to the single top eigenvector in the fully sampled case. In contrast, [Arora et al. \(2013\)](#) estimates the best  $d$ -dimensional subspace in the fully sampled case and provides a global convergence result by relaxing the non-convex problem to a convex one. We seek to identify the  $d$  dimensional subspace by solving the non-convex problem directly.

Our work is very closely related to [Balzano and Wright \(2014\)](#) and [Zhang and Balzano \(2015\)](#). In [Balzano and Wright \(2014\)](#), the authors prove that, within a local region of the true subspace, an expected improvement of their defined convergence metric for each iteration of GROUSE can be obtained. In contrast, we focus on global convergence results. We establish global convergence results from any random initialization for fully sampled data, and with compressively sampled data we provide a result on expected improvement on the defined convergence metric for each iteration. With missing data we provide expected improvement results on our defined convergence metric only within a local region of the true subspace, though our region is much less conservative than that required by [Balzano and Wright \(2014\)](#) and we provide a much simpler analysis framework that can be applied to different sampling strategies. Moreover, for each iteration of the GROUSE algorithm, the expected improvement on the convergence metric defined in [Balzano and Wright \(2014\)](#) only holds locally in both theory and practice, while our theoretical result provides a tighter bound for the global convergence behavior of GROUSE over a variety of simulations. This suggests that our result has more promise to be extended to a global result for missing data. In [Zhang and Balzano \(2015\)](#), the authors only consider fully sampled data and study the global convergence behavior of GROUSE in terms of two phases with two different convergence metrics. In comparison, we extend the results to undersampled data and provide an unifying analysis framework with only one single convergence metric for both fully sampled and undersampled data. With this much simpler analysis framework, the global convergence result of full data case we provide in this paper is still slightly tighter than that in [Zhang and Balzano \(2015\)](#). Hence the focus of this paper is more general and the analysis is more concise than that in [Zhang and Balzano \(2015\)](#).

Turning to batch methods, [R.H.Keshavan \(2012\)](#); [Jain et al. \(2013\)](#) provided the first theoretical guarantee for an alternating minimization algorithm for low-rank matrix recovery in the undersampled case. Under typical assumptions required for the matrix recovery problems [Recht et al. \(2010\)](#), they established geometric convergence to the global optimal solution. Earlier work [Keshavan et al. \(2010\)](#); [Ngo and Saad \(2012\)](#) considered the same undersampled problem formulation and established convergence guarantees for a steepest descent method (and a preconditioned version) on the full gradient, performed on the Grassmannian. [Chen and Wainwright \(2015\)](#); [Bhojanapalli et al. \(2015\)](#); [Zheng and Lafferty \(2015\)](#) considered low rank semidefinite matrix estimation problems, where they reparameterized the underlying matrix as  $M = UU^T$ , and update  $U$  via a first order gradient descent method. However, all these results require batch processing and a decent initialization that is close enough to the optimal point, resulting in a heavy computational burden and precluding problems with streaming data. We study random initialization, and our algorithm has fast, computationally efficient updates that can be performed in an online context.

Lastly, several convergence results for optimization on general Riemannian manifolds, including several special cases for the Grassmannian, can be found in [Absil et al. \(2009\)](#). Most of the results are very general; they include global convergence rates to local optima for steepest descent, conjugate gradient, and trust region methods, to name a few. We instead focus on solving the problem in (1) and provide global convergence rates to the global minimum.

### 3. Convergence Analysis

We analyze Algorithm 1, and derive results for both fully sampled data, *i.e.*,  $A_t = \mathbb{I}_n$ , and undersampled data, *i.e.*,  $A_t \in \mathbb{R}^{m \times n} (m \ll n)$ . At each step, the algorithm receives a vector  $x_t = A_t v_t$ , then forms the gradient of  $\min_w \|A_t U w - x_t\|_2^2$ , and takes a step in the direction of the negative

gradient restricted on the Grassmannian. More specifically, the algorithm works as follows: First we project our data vector onto a linear measured version of the current subspace iterate to get the exact (when  $A_t = \mathbb{I}_n$ ) or approximated projection  $p_t$  and residual  $r_t$ . Then update the iterates with a rank-one step as can be seen in Equation (4), which includes the new direction  $r_t$  while maintaining the orthonormality of the columns of  $U_{t+1}$  as well, thereby moving to a new point on the Grassmannian. This update is derived and explained in further detail in Balzano et al. (2010a); Edelman et al. (1998). The rank-one update tilts  $U_t$  to no longer contain  $p_t$  but instead contain a linear combination of  $p_t$  and  $r_t$ . As we will show in the following section, under mild conditions, the update will move our current estimate  $R(U_t)$  toward the true subspace either deterministically or expectedly.

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**Algorithm 1** GROUSE: Grassmannian Rank-One Update Subspace Estimation
 

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Given  $U_0$ , an  $n \times d$  matrix with orthonormal columns, with  $0 < d < n$ ;

Set  $t := 0$ ;

**repeat**

    Given sampling matrix  $A_t : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and observation  $x_t = A_t v_t$ ;

    Define  $w_t := \arg \min_w \|A_t U_t w - x_t\|_2^2$ ;

    Define  $p_t := U_t w_t$  and  $\tilde{r}_t := x_t - A_t p_t$ ,  $r_t := A_t^T \tilde{r}_t$ ;

    Using step size

$$\theta_t = \arctan \left( \frac{\|r_t\|}{\|p_t\|} \right) \quad (3)$$

    update with a gradient step on the Grassmannian:

$$U_{t+1} := U_t + \left( \frac{y_t}{\|y_t\|} - \frac{p_t}{\|p_t\|} \right) \frac{w_t^T}{\|w_t\|} \quad (4)$$

    where

$$\frac{y_t}{\|y_t\|} = \frac{p_t}{\|p_t\|} \cos(\theta_t) + \frac{r_t}{\|r_t\|} \sin(\theta_t)$$

$t := t + 1$ ;

**until** termination

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Before we present our main analysis, we first define our convergence metric and call out the main assumption on the underlying data.

**Definition 1 (Principal Angles)** We use  $\phi_{t,k}$ ,  $k = 1, \dots, d$  denote the principal angles between subspaces  $R(U_t)$  and  $R(\bar{U})$ , which are defined [Stewart and Sun (1990), Chapter 5] by  $\cos \phi_{t,k} = \sigma_{t,k}(\bar{U}^T U_t)$  with  $\sigma_{t,k}$  denoting the  $k^{\text{th}}$  singular value of  $\bar{U}^T U_t$ .

**Definition 2 (Determinant similarity)** Our measure of similarity between  $R(U_t)$  and  $R(\bar{U})$  is  $\zeta_t \in [0, 1]$ , defined as

$$\zeta_t := \det(\bar{U}^T U_t U_t^T \bar{U}) = \prod_{k=1}^d \cos^2 \phi_{t,k}.$$

The convergence metric  $\zeta_t$  increases to one when our estimation  $R(U_t)$  converges to  $R(\bar{U})$ , i.e., all principal angles between the two subspaces equal zero. Compared to the typical convergence metrics defined either as  $\|(I - \bar{U}\bar{U}^T)U_t\|_F^2 = d - \|\bar{U}^T U_t\|_F^2 = \sum_{k=1}^d \sin^2 \phi_{t,k}$  or  $1 - \|\bar{U}^T U_t\|_2^2 = \sin^2 \phi_{t,1}$ , our convergence metric  $\zeta_t$  measures the similarity instead of the discrepancy between  $R(U_t)$  and  $R(\bar{U})$ . In other words,  $\zeta_t$  achieves its maximum value one when  $R(U_t)$  converges to

$R(\bar{U})$ , while the typical subspace distance is zero when the subspaces are equal. Also note that  $\zeta_t = 0$  iff at least one of the principal angles is a right angle. That is, all stationary points of the full data problem  $U_{stat}$  except the true subspace have  $\det(\bar{U}^T U_{stat} U_{stat}^T \bar{U}) = 0$  Balzano (2012).

**Condition 1** *For the underlying data  $v_t = \bar{U} s_t$ , we assume the entries of  $s_t$  are uncorrelated and each has zero mean and unit variance.*

Now we are ready to present our main results on the convergence of the GROUSE algorithm. For simplicity of notation, we will drop the subscripts except  $\zeta_t$  hereafter.

### 3.1. Fully Sampled Noiseless Data

In this section, we consider fully sampled data. We first derive a greedy step size scheme for each iteration  $t$  that maximizes the improvement on our convergence metric  $\zeta_t$ . For each update we can prove the following (Appendix B):

$$\frac{\zeta_{t+1}}{\zeta_t} = \left( \cos \theta + \frac{\|v_\perp\|}{\|v_\parallel\|} \sin \theta \right)^2.$$

It follows that

$$\theta^* = \arg \max_{\theta} \frac{\zeta_{t+1}}{\zeta_t} = \arctan \left( \frac{\|v_\perp\|}{\|v_\parallel\|} \right).$$

This is equivalent to (3) in the fully sampled setting  $A_t = \mathbb{I}_n$ . Using  $\theta^*$ , we obtain monotonic improvement on the determinant similarity that can be quantified by the following lemma.

**Lemma 3 (Monotonic result for fully sampled noiseless case)** *For fully sampled data, choosing step size  $\theta^* = \arctan \left( \frac{\|v_\perp\|}{\|v_\parallel\|} \right)$ , after one iteration of GROUSE we obtain*

$$\frac{\zeta_{t+1}}{\zeta_t} = 1 + \frac{\|v_\perp\|^2}{\|v_\parallel\|^2} \geq 1$$

Under the mild assumption that each data vector is randomly sampled from the underlying subspace, we obtain strict improvement on  $\zeta_t$  for each iteration provided  $\|v_\perp\| > 0$  and  $\|v_\parallel\| > 0$ ; we discuss this further below. When we further consider apply Condition 1, we quantify the expected improvement at every step as follows.

**Lemma 4** *For each iteration of GROUSE, we have*

$$\mathbb{E} \left[ \frac{\|v_\perp\|^2}{\|v_\parallel\|^2} \middle| U \right] \geq \mathbb{E} \left[ \frac{\|v_\perp\|^2}{\|v\|^2} \middle| U \right] \geq \frac{1 - \zeta_t}{d}$$

Together with Lemma 3, we obtain the following lower bound on the expected improvement on  $\zeta_t$  for each iteration.

**Corollary 5 (Expected improvement on  $\zeta_t$ )** *When fully sampled data satisfying Condition 1 are input to the GROUSE algorithm, given the step size in Eq (3), the expected improvement after one update is given as:*

$$\mathbb{E} [\zeta_{t+1} | U] \geq \left(1 + \frac{1 - \zeta_t}{d}\right) \zeta_t$$

The above results provide insight into how the GROUSE algorithm converges to the global minimum of a non-convex problem formulation: GROUSE is not attracted to stationary points that are not the global minimum. As we mentioned previously, all other stationary points  $U_{stat}$  have  $\det(\bar{U}^T U_{stat} U_{stat}^T \bar{U}) = 0$ , because they have at least one direction orthogonal to  $\bar{U}$  Balzano (2012). Therefore, if the initial point  $U_0$  has determinant similarity with  $\bar{U}$  strictly greater than zero, then we are guaranteed to stay away from other stationary points, since GROUSE increases the determinant similarity monotonically, according to Lemma 3. We therefore may initialize GROUSE with  $U_0$  drawn uniformly from the Grassmannian, e.g., as the orthonormal basis of a random matrix  $V \in R^{n \times d}$  with entries being independent standard Gaussian variables, which guarantees  $\zeta_0 > 0$  with probability one.

**Lemma 6** *Nguyen et al. (2014) Initialize the starting point  $U_0$  of GROUSE as the orthonormalization of an  $n \times d$  matrix with entries being standard normal variables. Then*

$$\mathbb{E}[\zeta_0] = \mathbb{E} [\det(U_0^T \bar{U} \bar{U}^T U_0)] = C \left(\frac{d}{ne}\right)^d$$

where  $C > 0$  is a constant.

Finally, with Lemma 6 and Corollary 5 providing the expectation of initial value  $\mathbb{E}[\zeta_0]$  and the expected convergence rate of  $\zeta_t$ , we establish the global convergence result of GROUSE.

**Theorem 7 (Global Convergence of GROUSE)** *Let  $1 \geq \zeta^* > 0$  be the desired accuracy of our estimated subspace. With the initialization ( $U_0$ ) of GROUSE as the range of an  $n \times d$  matrix with entries being i.i.d standard normal random variables, then for any  $\rho > 0$ , after*

$$\begin{aligned} K &\geq K_1 + K_2 \\ &= \left(\frac{2d^2}{\rho} + 1\right) \tau_0 \log(n) + 2d \log\left(\frac{1}{2\rho(1 - \zeta^*)}\right) \end{aligned}$$

iterations of GROUSE Algorithm 1,

$$\mathbb{P}(\zeta_K \geq \zeta^*) \geq 1 - 2\rho.$$

where  $\tau_0 = 1 + \frac{\log\left(\frac{(1-\rho/2)}{C}\right) + d \log(e/d)}{d \log n}$  with  $C > 0$  is a constant approximately equal to 1.

The proof is provided in Appendix B, where we show that the iteration complexity is a combination of iterations required by two phases:  $K_1 = \left(\frac{2d^2}{\rho} + 1\right) \tau_0 \log(n)$  is the number of iterations required by GROUSE to achieve  $\zeta_t \geq 1/2$  from a random initialization  $U_0$ ; and  $K_2 = 2d \log\left(\frac{1}{2\rho(1 - \zeta^*)}\right)$  is the additional iterations required by GROUSE to converge to the given accuracy  $\zeta^*$  from  $\zeta_{K_1} =$



1/2. In [Zhang and Balzano \(2015\)](#), the required iterations for convergence to a given accuracy is  $O(d^3 \log(n)/\rho)$ . The authors also use a two phase analysis strategy with a different convergence metric in each phase. By leveraging the relationship between the two convergence metrics, they combine the convergence result in each phase to give the global one. Compared with [Zhang and Balzano \(2015\)](#), our result is slightly tighter while the analysis is much more concise.

#### 4. Compressively Sampled Data

In this section, we consider compressively sampled data where each vector  $v$  is subsampled by a sampling matrix  $A \in \mathbb{R}^{m \times n}$  ( $m \ll n$ ), i.e., the number of measurements is much smaller than the ambient dimension. We use an approach that merges linear algebra with random matrix theory to establish the improvement on  $\zeta_t$  in terms of expectation for each iteration. We show that with Gaussian sampling matrices, the determinant similarity increases in expectation with a rate similar to that of fully sampled case, roughly slowed by the compression ratio ( $m/n$ ) of our sampling matrices.

**Theorem 8** *Suppose each sampling matrix  $A_t$  has i.i.d Gaussian entries distributed as  $\mathcal{N}(0, 1/n)$ , and further suppose  $m \geq C_1 d \log n$ . Then with probability exceeding  $1 - n^{-C_1 \delta_1^2 d/2} - n^{-C_1 \delta_2^2 d/2}$  we obtain*

$$\mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} \middle| U \right] \geq 1 + \eta_1 \frac{m}{n} \frac{1 - \zeta_t}{d}$$

where  $\eta_1 = \frac{1 - \frac{m+1}{n} - 2 \frac{d}{m}}{(1 + \eta_2)^2}$  and  $\eta_2 = \frac{\sqrt{d/m + \delta_2} \sqrt{d/m}}{1 - \delta_1 - \sqrt{d/m}}$  with  $0 < \delta_1 < 1 - \sqrt{d/m}$  and  $\delta_2 > 0$ .

This theorem implies that, for each iteration of GROUSE, expected improvement on  $\zeta_t$  can be obtained with high probability as long as the number of samples is  $O(d \log n)$ . In the high dimensional setting, compared to the fully sampled data case, the expected improvement on  $\zeta_t$  is approximately scaled down by the sampling density,  $m/n$ . However, due to the uncertainty induced by the compressed sampling framework, the improvement on the determinant similarity is not monotonic, which is the hurdle to pass before we can provide a global convergence result similar to Theorem 7 for fully sampled data. By leveraging techniques in stochastic process theory, it may be possible to establish asymptotic convergence results or even non-asymptotic convergence results in terms of the number of iterations required before GROUSE first achieves a given accuracy. We leave this as future work.

Now we provide the following intermediate results that allow us to establish Theorem 8, the proof of which can be found in Appendix C. We first call out Lemma 9 that quantifies the improvement on  $\zeta_t$ .

**Lemma 9** *Suppose  $AU$  has full column rank, then for each iteration of GROUSE we have*

$$\frac{\zeta_{t+1}}{\zeta_t} \geq 1 + \frac{2 \|\tilde{r}\|^2 - \|r\|^2}{\|p\|^2} + 2 \frac{\Delta}{\|p\|^2}$$

where  $\Delta = w_2^T (\bar{U}^T U)^{-1} \bar{U}^T r$  with  $w_2 = (U^T A^T A U)^{-1} U^T A^T A v_\perp$ .

Compared to the fully sampled case (Lemma 3) where  $\Delta = 0$  and  $\|\tilde{r}\|^2 = \|r\|^2$ , this result highlights the perturbation induced by the compressed sampling framework. However, as we present



in Lemma 10, we can prove that with Gaussian sampling matrices  $\mathbb{E}[\Delta] = 0$ . Then by leveraging random matrix theory, the length of projection ( $\|p_t\|$ ) and residual ( $\|\tilde{r}\|, \|r\|$ ) can be bounded either with high probability or in terms of expectation, which together with Lemma 9 and Corollary 5 complete the proof.

**Lemma 10** *With the same conditions as Theorem 8, then*

$$\mathbb{E} [\Delta | U, v] = 0$$

**Lemma 11** *With the same conditions as Theorem 8, then*

$$\mathbb{P} \left( \|p\|^2 \leq (1 + \eta_2)^2 \|v\|^2 \right) \geq 1 - n^{-C_1 \delta_1^2 d/2} - n^{-C_1 \delta_2^2 d/2}$$

where  $\eta_2 = \frac{\sqrt{d/m + \delta_2} \sqrt{d/m}}{1 - \delta_1 - \sqrt{d/m}}$  with  $0 < \delta_1 < 1 - \sqrt{d/m}$  and  $\delta_2 > 0$ .

**Lemma 12** *With the same conditions as Theorem 8, then*

$$\mathbb{E} [\|\tilde{r}\|^2 | v_\perp] = \frac{m}{n} \left( 1 - \frac{d}{m} \right) \|v_\perp\|^2$$

and

$$\mathbb{E} [\|r\|^2 | v_\perp] \leq \left( 1 + \frac{m+1}{n} \right) \frac{m}{n} \|v_\perp\|^2$$

Now we are ready to prove our main result.

**Proof of Theorem 8** **Proof** Given the above intermediate results and Lemma 4, with probability exceeding  $1 - n^{-C_1 \delta_1^2 d/2} - n^{-C_1 \delta_2^2 d/2}$  we have

$$\begin{aligned} \mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} \middle| U \right] &\geq 1 + \mathbb{E} \left[ \frac{2\|\tilde{r}\|^2 - \|r\|^2 + 2\Delta}{\|p\|^2} \middle| U \right] \\ &\geq 1 + \frac{1}{(1 + \eta_2)^2} \mathbb{E}_v \left[ \mathbb{E} \left[ \frac{2\|\tilde{r}\|^2 - \|r\|^2 + 2\Delta}{\|v\|^2} \middle| v, U \right] \right] \\ &\geq 1 + \frac{1}{(1 + \eta_2)^2} \mathbb{E}_v \left[ \mathbb{E} \left[ \frac{2\|\tilde{r}\|^2 - \|r\|^2}{\|v\|^2} \middle| v, U \right] \right] \\ &\geq 1 + \frac{1 - \frac{m+1}{n} - 2\frac{d}{m}}{(1 + \eta_2)^2} \frac{m}{n} \mathbb{E}_v \left[ \frac{\|v_\perp\|^2}{\|v\|^2} \middle| U \right] \\ &\geq 1 + \frac{1 - \frac{m+1}{n} - 2\frac{d}{m}}{(1 + \eta_2)^2} \frac{m}{n} \frac{1 - \zeta_t}{d} \end{aligned}$$

■

## 5. Missing Data

In this section, we study the convergence of GROUSE for the missing data case. We show that within the local region of the true subspace, we obtain an expected monotonic improvement on our defined convergence metric with high probability. For consistency with literature, we use  $\Omega$  to denote the indices of observed entries for each data vector, *i.e.*,  $Av = v_\Omega, AU = U_\Omega$ . In this paper, we assume  $\Omega$  are uniformly sampled over  $\{1, 2, \dots, n\}$  with replacement. In other words, we assume each row of the sampling matrices  $A$  is uniformly sampled from the rows of identity matrix  $\mathbb{I}_n$  with replacement.

Before we present our main results, we first call out the typical incoherence assumption on the underlying data.

**Condition 2** *A subspace  $R(U)$  is incoherent with parameter  $\mu$  if*

$$\max_{i \in \{1, \dots, n\}} \|\mathcal{P}_U e_i\|_2^2 \leq \frac{\mu d}{n}$$

where  $e_i$  is the  $i^{\text{th}}$  canonical basis vector.

Note that  $1 \leq \mu \leq \frac{n}{d}$ . According to the above definition, the incoherence parameter of a vector  $z \in \mathbb{R}^n$  can be defined as:

$$\mu(z) = \frac{n \|z\|_\infty}{\|z\|_2^2}$$

In this section, we assume the true subspace  $R(\bar{U})$  is incoherent with parameter  $\mu_0$ , and use  $\mu(U)$ ,  $\mu(v_\perp)$  to denote the incoherence parameter of  $\mathbb{R}(U)$  and  $v_\perp$ . We now show the expected improvement of  $\zeta_t$  in a local region of the true subspace.

**Theorem 13** *Suppose  $\sum_{k=1}^d \sin^2 \phi_k \leq \frac{d\mu_0}{16n}$ . If  $m \geq \frac{32}{3} d\mu_0 \log(n\sqrt{2d})$ , then with probability at least  $1 - \frac{3}{n^2}$  we have*

$$\mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} \middle| U \right] \geq 1 + \eta_0 \frac{m}{n} \frac{1 - \zeta_t}{d}$$

where  $\eta_0 = \frac{1}{(1+\eta_3)^2} \left( 1 - \frac{2\mu_0}{1-\gamma} \frac{d}{m} - \frac{2\gamma_1\eta_3}{\sqrt{1-d\mu_0/16n}} \right)$  with  $\eta_3 = \frac{1+2\sqrt{\mu(v_\perp) \log(n)}}{1-\gamma} \sqrt{\frac{2d\mu_0}{m}}$ ,

$\gamma = \sqrt{\frac{32d\mu_0}{3m} \log(n\sqrt{2d})}$  and  $\gamma_1 = \sqrt{\frac{d\mu_0}{16n} + \frac{d\mu_0}{mn}}$ .

This theorem shows that, within the local region of the true subspace, expected improvement on  $\zeta_t$  can be obtained with high probability as long as the sampling number is on the order of  $O(d \log(n\sqrt{d}))$ . Comparing with the Gaussian sampling framework, here we can only obtain expected improvement within the local region of the true subspace. However, as we demonstrate in Section 6, this local region is too conservative, and the lower bound in Theorem 13 holds from a random initialization as in the Gaussian sampling framework. This gap is induced by the challenge of maintaining the incoherence property of our estimates  $R(U)$ , for which we had to consider the worst case. We leave the extension of the local convergence results to the global one as future work.

In order to compare to Balzano and Wright (2014), consider the following corollary.

**Corollary 14** Define the determinant discrepancy as  $\kappa_t = 1 - \zeta_t$ , then under the same conditions as Theorem 13, we have

$$\mathbb{E} [\kappa_{t+1} | \kappa_t] \leq \left( 1 - \eta_0 \left( 1 - \frac{d\mu_0}{16n} \right) \frac{m}{nd} \right) \kappa_t$$

with probability exceeding  $1 - 3/n^2$ .

**Proof** Let  $X = [X_1, \dots, X_d]$  with  $X_i = \sin^2 \phi_{t,i}$ . Let  $f(X) = 1 - \sum_{i=1}^d X_i - \Pi_{i=1}^d (1 - X_i)$ , then  $\frac{\partial f(X)}{\partial X_i} = -1 + \Pi_{j \neq i} (1 - X_j) \leq 0$ . That is,  $f(X)$  is a decreasing function of each component. Therefore,  $f(X) \leq f(0) = 0$ . It follows that

$$\zeta_t = \Pi_{i=1}^d (1 - X_i) \geq 1 - \sum_{i=1}^d X_i \geq 1 - \frac{d\mu_0}{16n} \quad (5)$$

With a slight modification of Theorem 13 we obtain

$$\mathbb{E} [\kappa_{t+1} | \kappa_t] \leq \left( 1 - \eta_0 \frac{m}{n} \frac{\zeta_t}{d} \right) \kappa_t. \quad (6)$$

(5) and (6) together complete the proof. ■

Recall that  $1 \leq \mu_0 \leq \frac{n}{d}$ , therefore the expected linear decay rate of  $\kappa_t$  is at least  $1 - (1 - 1/16)\eta_0 \frac{m}{nd}$ . In Balzano and Wright (2014) (Corollary 2.15), a similar linear convergence result is established in terms of the Frobenius norm discrepancy between  $R(\bar{U})$  and  $R(U)$ , denoted as  $\epsilon_t = \sum_{i=1}^d \sin^2 \phi_{t,d}$ . However, their result only holds when  $\epsilon_t \leq (8 \times 10^{-6}) \frac{m}{n^3 d^2}$  which is more conservative than ours. Moreover, as we mentioned previously, empirical evidence shows the lower bound in Theorem 13 holds for every iteration from any random initialization. In contrast, in Balzano and Wright (2014), even for numerical results expected linear improvements only hold within the local region of the true subspace.

Now we present the following intermediate results for the proof of Theorem 13. First, we show improvement on  $\zeta_t$  equivalent to Lemma 9.

**Lemma 15** Assume  $U_\Omega$  has full column rank, then

$$\frac{\zeta_{t+1}}{\zeta_t} \geq 1 + \frac{\|r_\Omega\|^2}{\|p\|^2} + 2 \frac{\Delta}{\|p\|^2}$$

where  $\Delta = w_2^T (\bar{U}^T U)^{-1} \bar{U}_\Omega^T r_\Omega$  with  $w_2 = (U_\Omega^T U_\Omega)^{-1} U_\Omega^T v_{\perp, \Omega}$ .

Next we establish concentration results on the key relevant quantities.

**Lemma 16** Let  $\delta > 0$  and  $m \geq \frac{8}{3} d\mu(U) \log(2d/\delta)$ , then with probability at least  $1 - \delta$ ,

$$\mathbb{E} [\|r_\Omega\|^2 | v_\perp] \geq \left( 1 - \frac{\mu(U)}{1 - \gamma'} \frac{d}{m} \right) \frac{m}{n} \|v_\perp\|^2$$

where  $\gamma' = \sqrt{\frac{8d\mu(U)}{3m} \log(\frac{2d}{\delta})}$ .

**Lemma 17** Under the same condition as Lemma 16, then with probability at least  $1 - 2\delta$  we have

$$\|p\|^2 \leq (1 + \eta'_3)^2 \|v\|^2$$

where  $\eta'_3 = \frac{1 + \sqrt{2\mu(v_\perp) \log(1/\delta)}}{1 - \gamma'} \sqrt{\frac{d\mu(U)}{m}}$  with  $\gamma'$  be the same as Lemma 16.

**Lemma 18** With probability at least  $1 - 3\delta$  we have

$$\mathbb{E} [\Delta | v_\perp] \leq \frac{\gamma'_1 \eta'_3}{\cos \phi_d} \frac{m}{n} \|v_\perp\|^2$$

where  $\gamma'_1 = \sqrt{\sin^2 \phi_d + \frac{d\mu_0}{mn}}$  and  $\eta'_3$  be the same as Lemma 16, 17.

**Lemma 19** Balzano and Wright (2014) Suppose  $\sum_{i=1}^d \sin^2 \phi_k \leq \frac{d}{16n} \mu_0$ , then  $\mu(U) \leq 2\mu_0$ .

Now we are ready to prove Theorem 13.

**Proof of Theorem 13** **Proof** Given the condition required by Theorem 13, we have  $\sin \phi_d \leq \sqrt{d\mu_0/16n}$  and  $\cos \phi_d \geq \sqrt{1 - d\mu_0/16n}$ . Let  $\eta_4 = 1 - \frac{2\mu_0}{1-\gamma} \frac{d}{m}$  and  $\eta_5 = \frac{\gamma_1 \eta_3}{\sqrt{1 - d\mu_0/16n}}$ , then Lemma 15 to Lemma 19 together yield the following:

$$\begin{aligned} \mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} | U \right] &\geq 1 + \frac{1}{(1 + \eta_3)^2} \mathbb{E} \left[ \mathbb{E} \left( \frac{\|r_\Omega\|^2 + 2\Delta}{\|v\|^2} \middle| U, v \right) \right] \\ &\geq 1 + \frac{\eta_4 - 2\eta_5}{(1 + \eta_3)^2} \frac{m}{n} \mathbb{E} \left[ \frac{\|v_\perp\|^2}{\|v\|^2} \middle| U, v \right] \\ &\geq 1 + \frac{\eta_4 - 2\eta_5}{(1 + \eta_3)^2} \frac{m}{n} \frac{1 - \zeta_t}{d} \end{aligned}$$

The probability bound is obtained by taking the union bound of those generating Lemma 16 to Lemma 19, as we can see in the proofs of them in Appendix D this union bound is at least  $1 - 3\delta$ . Then choosing  $\delta$  to be  $1/n^2$  completes the proof.  $\blacksquare$

## 6. Numerical Results

In this section, we demonstrate that our theoretical results match the empirical convergence behavior of GROUSE. We generate the underlying data matrix  $M = [v_1, v_2, \dots, v_T]$  as  $M = \bar{U}W$ . For both the fully sampled data case and compressively sampled data case, the underlying signals are generated from a sparse subspace, demonstrating that incoherence assumptions are not required by our results for these two cases. Specifically, the underlying subspace of each trial is set to be a sparse subspace, as the range of an  $n \times d$  matrix  $\bar{U}$  with sparsity on the order of  $\frac{\log(n)}{n}$ . For the missing data case, we generate the underlying subspace as the range of  $n \times d$  matrix with i.i.d standard normal distribution. The entries of the coefficient matrix  $W$  for all three cases are generated as i.i.d  $\mathcal{N}(0, 1)$  that satisfies Condition 1. We also want to mention that we run GROUSE with random initialization for all of the plots in this section.

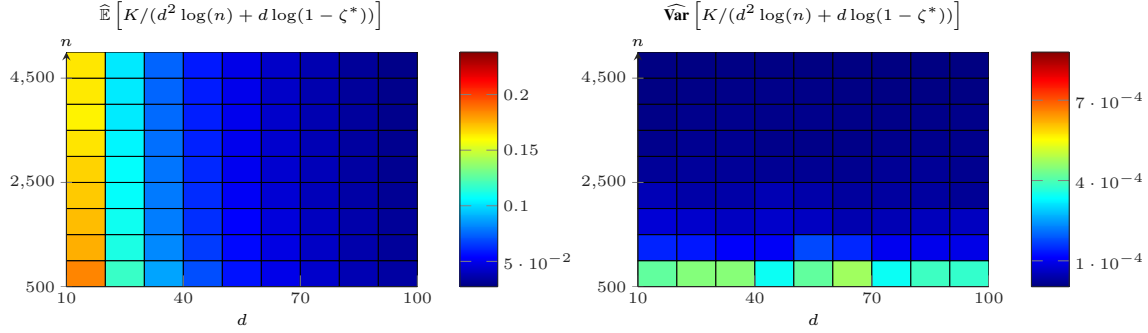


Figure 1: Illustration of the bounds on  $K$  compared to their values in practice, averaged over 50 trials with different  $n$  and  $d$ . We show the ratio of  $K$  to the bound  $d^2 \log(n) + d \log(1 - \zeta^*)$ .

We first examine our global convergence result, *i.e.*, Theorem 7, for the fully sampled data in Figure 1. We run GROUSE to convergence for a required accuracy  $\zeta^* = 1 - 1e-4$  and show the ratio of  $K$  to the bound described in Theorem 7,  $d^2 \log(n) + d \log \frac{1}{1-\zeta^*}$ . We run GROUSE over 50 trials and show the mean and variance. We can see that, for fixed  $n$ , our theoretical results become more and more loose as we increase the dimension of the underlying subspace. However, compared to the empirical mean, the empirical variance is very small. This indicates that the relationship between our theoretical upper bounds and the actual iterations required by GROUSE is stable.

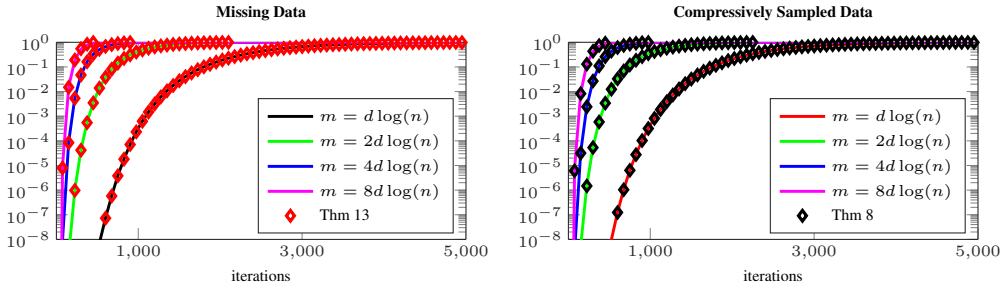


Figure 2: Illustration of expected convergence bounds given by Theorem 13 (left) and Theorem 8 (right) over 50 trials. We set  $n = 5000$ ,  $d = 10$ . The diamonds denote the lower bound on expected convergence rates described in Theorem 13 and Theorem 8.

Next we examine our theoretical results (Theorem 8, 13) for the expected improvement on  $\zeta_t$  for the undersampled case in Figure 2. We set  $n = 5000$  and  $d = 10$ . We run GROUSE over different sampling numbers  $m$ . The plots are obtained by averaging over 50 trials. We can see that our theoretical bounds on the expected improvement on  $\zeta_t$  for both missing data and compressively sampled data are tight from any random initialization, although we have only established local convergence results for the missing data case. Also note that Theorem 8 and Theorem 13 indicate that the expected improvement on the determinant similarity has a similar form to that of the fully sampled case scaled by the sampling density ( $m/n$ ). These together motivate us to approximate the

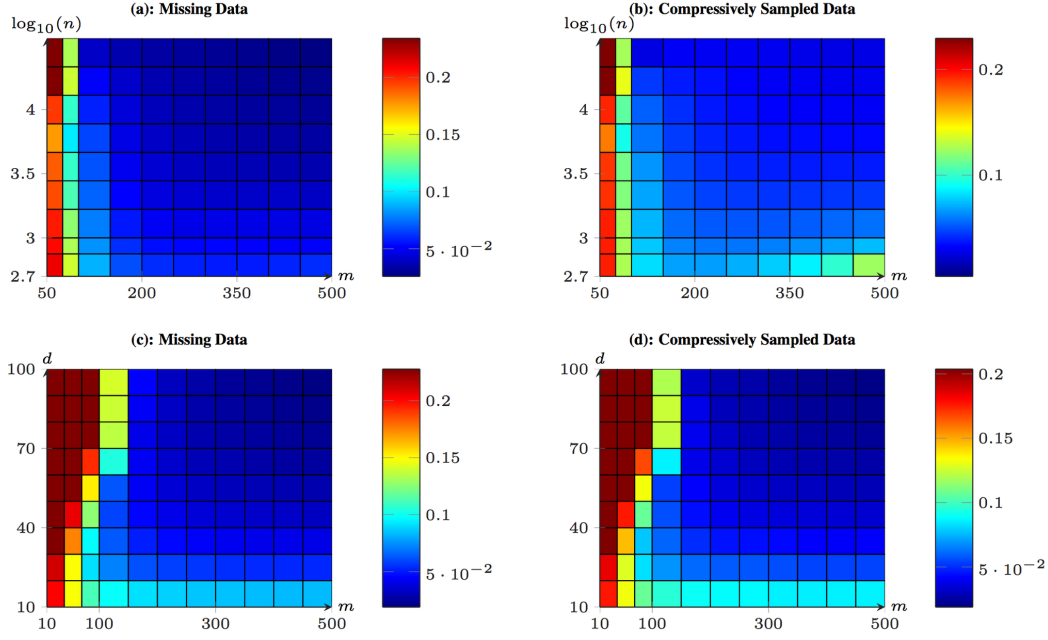


Figure 3: Illustration of our heuristic bounds on  $K$  (the actual iterations required by GROUSE to converge to the given accuracy) over different  $d$ ,  $m$  and  $n$ , averaged over 20 trials. In this simulation, we run GROUSE from a random initialization to convergence for a required accuracy  $\zeta^* = 1 - 1e-3$ . We show the ratio of  $K$  to the heuristic bound  $\frac{n}{m} (d^2 \log(n) + d \log(1 - \zeta^*))$ . In (a) and (b), we set  $d = 50$  and examine  $K$  over  $m$  and  $n$  for both *missing data* (a) and *compressively sampled data* (b). In (c) and (d), we set  $n = 10000$  and examine  $K$  over  $m$  and  $d$  for both *missing data* (c) and *compressively sampled data* (d). In these plots, we use the dark red to indicate the failure of convergence.

required iterations to achieve a given accuracy as that required by the fully sampled case times the reciprocal of sampling density,  $n/m$ :

$$\frac{n}{m} (d^2 \log(n) + d \log(1 - \zeta^*)) .$$

As we see in Figure 3, the ratio of the our heuristic matches the simulated data well. That is, when  $m$  is slightly larger than  $d$ , the empirical mean of the ratio of the actual iterations required by GROUSE to our heuristic bound is similar to that of the full data case. We leave the rigorous proof of this heuristic as future work.

## 7. Conclusion

We analyzed a manifold gradient descent algorithm applied to a non-convex optimization formulation for recovering a low-dimensional subspace model from streaming data from that subspace. We provide a simplified analysis as compared to Zhang and Balzano (2015), showing global convergence for an incremental gradient descent method on the Grassmannian for fully sampled data.

For optimizing a particular cost function (1) in this case, we show that with probability exceeding  $1 - 2\rho$ , the gradient algorithm converges from any random initialization to any desired accuracy of  $\zeta^*$  within  $O\left(\frac{d^2 \log(n)}{\rho} + d \log\left(\frac{1}{2\rho(1-\zeta^*)}\right)\right)$  iterations.

With undersampled data, we show that expected improvement on our convergence metric  $\zeta_t$  can be obtained with high probability for each iteration  $t$ . We prove that, comparing with fully sampled data, the expected improvement on determinant similarity is roughly proportional to the sampling density. With compressively sampled data this expected improvement holds from any random initialization, while it only holds within the local region of the true subspace for the missing data case. Establishing the global convergence result similar to that of fully sampled data remains as future work.

## Appendix A. Preliminaries

We start by providing the following lemma that we will use regularly in the manipulation of the matrix  $\bar{U}^T U$ . It also provides us with more insights into our metric of determinant similarity between the subspaces. The proof can be found in [Stewart and Sun \(1990\)](#).

**Lemma 20** ([Stewart and Sun \(1990\)](#), [Theorem 5.2](#)) *There are unitary matrices  $Q$ ,  $\bar{Y}$ , and  $Y$  such that*

$$Q\bar{U}\bar{Y} := \begin{matrix} & d \\ d & \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \\ n-2d & \end{matrix} \text{ and } QUY := \begin{matrix} & d \\ d & \begin{pmatrix} \Gamma \\ \Sigma \\ 0 \end{pmatrix} \\ n-2d & \end{matrix}$$

where  $\Gamma = \text{diag}(\cos \phi_{t,1}, \dots, \cos \phi_{t,d})$ ,  $\Sigma = \text{diag}(\sin \phi_{t,1}, \dots, \sin \phi_{t,d})$  with  $\phi_{t,i}$  being the  $i^{\text{th}}$  principal angle between  $R(U)$  and  $R(\bar{U})$  defined in [Definition 1](#).

Now we are going to prove [Lemma 4](#) which is essential for us to establish the expected improvement on the determinant similarity for each iteration under different cases. Before that, we present the following lemmas which are crucial for the proof.

**Lemma 21** ([Balzano and Wright \(2014\)](#), [Lemma 2.12](#)) *Given any matrix  $Q \in \mathbb{R}^{d \times d}$  suppose that  $x \in \mathbb{R}^d$  is a random vector with entries identically distributed, zero-mean, and uncorrelated, then*

$$\mathbb{E} \left[ \frac{x^T Q x}{x^T x} \right] = \frac{1}{d} \text{tr}(Q)$$

**Lemma 22** [De Sa et al. \(2015\)](#) *Let  $X = [X_1, \dots, X_d]$  with  $X_i \in [0, 1]$ ,  $i = 1, \dots, d$ , then*

$$d - \sum_{i=1}^d X_i \geq 1 - \prod_{i=1}^d X_i$$



**Proof of Lemma 4** **Proof** According to Lemma 21 and Lemma 22 we have the following

$$\begin{aligned}\mathbb{E} \left[ \frac{\|v_\perp\|^2}{\|v\|^2} \middle| U \right] &= \mathbb{E} \left[ \frac{\|\bar{U}s\|^2 - \|U U^T \bar{U}s\|^2}{\|\bar{U}s\|^2} \middle| U \right] \\ &\stackrel{\vartheta_1}{=} \mathbb{E} \left[ \frac{s^T \bar{Y} (I - \Gamma^2) \bar{Y}^T s}{s^T s} \middle| U \right] \\ &= \frac{1}{d} \text{tr} (I - \Gamma^2) \stackrel{\vartheta_2}{\geq} \frac{1 - \zeta_t}{d}\end{aligned}\tag{7}$$

where  $\vartheta_1$  follows by  $\|\bar{U}s\|^2 = \|s\|^2$ , and  $\vartheta_2$  follows from Lemma 22 by setting  $X_i = \cos^2 \phi_{t,i}$ . ■

## Appendix B. Proof of Fully Sampled Data

**Proof of Lemma 3** **Proof** Let  $\frac{y}{\|y\|} = \cos(\theta) \frac{v_\parallel}{\|v_\parallel\|} + \sin(\theta) \frac{v_\perp}{\|v_\perp\|}$ , then according to (4) we have

$$\begin{aligned}\det(\bar{U}^T U_{t+1}) &= \det \left( \bar{U}^T U + \left( \frac{\bar{U}^T y}{\|y\|} - \frac{\bar{U}^T v_\parallel}{\|v_\parallel\|} \right) \frac{w^T}{\|w\|} \right) \\ &\stackrel{\vartheta_1}{=} \det(\bar{U}^T U) \left( 1 + \frac{w^T (\bar{U}^T U)^{-1}}{\|w\|} \left( \frac{\bar{U}^T y}{\|y\|} - \frac{\bar{U}^T v_\parallel}{\|v_\parallel\|} \right) \right) \\ &\stackrel{\vartheta_2}{=} \det(\bar{U}^T U) \frac{w^T (\bar{U}^T U)^{-1} \bar{U}^T y}{\|y\| \|w\|} \\ &\stackrel{\vartheta_3}{=} \det(\bar{U}^T U) \left( \cos \theta + \frac{\|v_\perp\|}{\|v_\parallel\|} \sin \theta \right) \\ &\stackrel{\vartheta_4}{=} \det(\bar{U}^T U) \frac{\|v\|}{\|v_\parallel\|}\end{aligned}$$

where  $\vartheta_1$  follows from the fact that for any invertible matrix  $M$  we have  $\det(M + ab^T) = \det(A) (1 + b^T M^{-1} a)$ ;  $\vartheta_2$  and  $\vartheta_3$  hold since  $\|v_\parallel\|^2 = \|Uw\|^2 = \|w\|^2$  and the following

$$w^T (\bar{U}^T U)^{-1} \bar{U}^T v_\parallel \stackrel{w=U^T \bar{U}s}{=} v^T v_\parallel = \|v_\parallel\|^2 \tag{8a}$$

$$w^T (\bar{U}^T U)^{-1} \bar{U}^T v_\perp \stackrel{w=U^T \bar{U}s}{=} v^T v_\perp = \|v_\perp\|^2; \tag{8b}$$

and  $\vartheta_4$  follows by replacing  $\theta = \arctan(\|v_\perp\|/\|v_\parallel\|)$ .

It therefore follows that  $\frac{\zeta_{t+1}}{\zeta_t} = \frac{\det(\bar{U}^T U_{t+1})^2}{\det(\bar{U}^T U)} = \frac{\|v\|^2}{\|v_\parallel\|^2} = 1 + \frac{\|v_\perp\|^2}{\|v_\parallel\|^2}$ . ■

**Proof of Theorem 7** **Proof** Let  $\kappa_t = 1 - \zeta_t$  denote the determinant *discrepancy* between  $R(\bar{U})$  and  $R(U)$ . According to Corollary 5 we have the following:

$$\mathbb{E} \left[ \frac{\zeta_{t+1}}{\zeta_t} \middle| U \right] \geq 1 + \frac{1 - \zeta_t}{d} \tag{9a}$$

$$\mathbb{E} \left[ \frac{\kappa_{t+1}}{\kappa_t} \middle| U \right] \leq 1 - \frac{1 - \kappa_t}{d} \tag{9b}$$

Therefore, the expected convergence rate of  $\zeta_t$  is faster when  $R(U)$  is far away from  $R(\bar{U})$ , while that of  $\kappa_t$  is faster when  $R(U)$  is close to  $R(\bar{U})$ . This motivates us to first use  $\zeta_t$  to get the necessary iterations for GROUSE converging to a local region of global optimal point from a random initialization. From there, we obtain the necessary iterations for GROUSE to converge to the required accuracy by leveraging (9b).

According to Lemma 3,  $\zeta_t$  is a non-decreasing sequence. Therefore, there exists  $T \geq 1$  such that  $\zeta_t \leq 1 - \frac{\rho}{2}$ ,  $\forall t \leq T$  where  $\rho \in (0, 1]$ . Then together with Corollary 5 we obtain the following: for any  $t \leq T$ ,

$$\mathbb{E} \left[ \zeta_{t+1} \middle| U \right] \geq \left( 1 + \frac{1 - \zeta_t}{d} \right) \zeta_t \geq \left( 1 + \frac{\rho}{2d} \right) \zeta_t .$$

Taking expectation of both sides, we obtain the following:

$$\mathbb{E} [\zeta_{t+1}] \geq \left( 1 + \frac{\rho}{2d} \right) \mathbb{E} [\zeta_t]$$

Therefore after  $K_1 \geq (2d/\rho + 1) \log \frac{1 - \frac{\rho}{2}}{\mathbb{E}[\zeta_0]}$  iterations of GROUSE we have

$$\begin{aligned} \mathbb{E} [\zeta_{K_1}] &\geq \left( 1 + \frac{\rho}{2d} \right)^{K_1} \mathbb{E} [\zeta_0] \\ &\geq \left( \left( 1 + \frac{\rho}{2d} \right)^{\frac{2d}{\rho} + 1} \right)^{\log \frac{1 - \frac{\rho}{2}}{\mathbb{E}[\zeta_0]}} \mathbb{E} [\zeta_0] \\ &\geq \mathbb{E} [\zeta_0] e^{\log \frac{1 - \frac{\rho}{2}}{\mathbb{E}[\zeta_0]}} = 1 - \frac{\rho}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P} \left( \zeta_{K_1} \geq \frac{1}{2} \right) &= 1 - \mathbb{P} \left( 1 - \zeta_{K_1} \geq \frac{1}{2} \right) \\ &\stackrel{\vartheta_1}{\geq} 1 - \frac{\mathbb{E}[1 - \zeta_{K_1}]}{1/2} \geq 1 - \rho \end{aligned} \tag{10}$$

where  $\vartheta_1$  follows by applying Markov inequality to the nonnegative random variable  $1 - \zeta_{K_1}$ . If  $T \leq K_1$ , then (10) automatically holds. Therefore, together with the following derived from Lemma 6, we obtain  $\log \left( \frac{1 - \rho/2}{\mathbb{E}[\zeta_0]} \right) = \log \left( \frac{1 - \rho/2}{C \left( \frac{d}{ne} \right)^d} \right) = \tau_0 d \log(n)$  with  $\tau_0 = 1 + \frac{\log \frac{(1 - \rho/2)}{C} + d \log(e/d)}{d \log n}$ .

Now with probability at least  $1 - \rho$ , for all  $t \geq K_1$  we have the following

$$\mathbb{E} [\kappa_{t+1} | U] \leq \left( 1 - \frac{1 - \kappa_t}{d} \right) \kappa_t \leq \left( 1 - \frac{1}{2d} \right) \kappa_t$$

Taking expectation of both sides, we have

$$\mathbb{E} [\kappa_{t+1}] \leq \left( 1 - \frac{1}{2d} \right) \mathbb{E} [\kappa_t]$$

After  $K_2 \geq 2d \log \frac{1/2}{\rho(1-\zeta^*)} \geq 2d \log \frac{\mathbb{E}[\eta_{K_1}]}{\rho(1-\zeta^*)}$  additional iterations of GROUSE we obtain

$$\begin{aligned} \mathbb{E}[\eta_{t+K_1}] &\leq \left(1 - \frac{1}{2d}\right)^{K_2} \mathbb{E}[\eta_{K_1}] \\ &\leq \left(1 - \frac{1}{2d}\right)^{2d \log \frac{\mathbb{E}[\eta_{K_1}]}{\rho(1-\zeta^*)}} \mathbb{E}[\eta_{K_1}] \\ &\leq \rho(1 - \zeta^*) \end{aligned}$$

Hence following similar argument as before we have

$$\begin{aligned} \mathbb{P}(\zeta_{K_1+K_2} \geq \zeta^*) &= 1 - \mathbb{P}(\eta_{K_1+K_2} \geq 1 - \zeta^*) \\ &\geq 1 - \frac{\mathbb{E}[\eta_{K_1+K_2}]}{1 - \zeta^*} \geq 1 - \rho \end{aligned} \tag{11}$$

(10) and (11) together complete the proof. ■

### Appendix C. Proof for Compressively Sampled Data

**Definition 23** We decompose  $w$  as  $w = w_1 + w_2$  where

$$w_1 = (U^T A^T A U)^{-1} U^T A^T A v_{\parallel} \quad w_2 = (U^T A^T A U)^{-1} U^T A^T A v_{\perp}$$

Let  $\tilde{v}_{\parallel} = U w_2$  denote the perturbation term induced by the undersampled framework. Note that in the fully sampled case, i.e.,  $A = \mathbb{I}_n$ , we have  $w_2 = 0$  and  $\|\tilde{v}_{\parallel}\| = 0$ .

With this definition, we have the following lemma.

**Lemma 24** Assume that  $AU$  has full column rank, then

$$w_1 = U^T v_{\parallel} = U^T v.$$

**Proof** Let  $a = U^T v_{\parallel}$ ;  $a$  is the unique solution to  $Uw = v_{\parallel}$ . Since  $AU$  also has full column rank,  $b = (U^T A^T A U)^{-1} U^T A^T A v_{\parallel}$  is the unique solution to  $AUw = Av_{\parallel}$ . However,  $AUa = Av_{\parallel}$  as well. Therefore,  $a = b$ . ■

Now we are ready to prove Lemma 9.

**Proof of Lemma 9** **Proof** Lemma 24 implies the following

$$w_1^T (\bar{U}^T U)^{-1} \bar{U}^T r = s^T \bar{U}^T U (\bar{U}^T U)^{-1} \bar{U}^T r = v^T A^T \tilde{r} = \|\tilde{r}\|^2$$

this together with  $w^T (\bar{U}^T U)^{-1} \bar{U}^T p = w^T (\bar{U}^T U)^{-1} \bar{U}^T U w = \|p\|^2$  yields

$$\begin{aligned} \det(\bar{U}^T U_{t+1}) &= \det\left(\bar{U}^T U + \bar{U}^T \left(\frac{p+r}{\|p+r\|} - \frac{p}{\|p\|}\right) \frac{w^T}{\|w\|}\right) \\ &\stackrel{\vartheta_1}{=} \det(\bar{U}^T U) \frac{w^T (\bar{U}^T U)^{-1} \bar{U}^T (p+r)}{\|p\| \sqrt{\|p\|^2 + \|r\|^2}} \\ &= \det(\bar{U}^T U) \frac{\|p\|^2 + \|r\|^2 + \Delta}{\|p\| \sqrt{\|p\|^2 + \|r\|^2}} \end{aligned}$$

where  $\vartheta_1$  holds since for any invertible matrix  $M$  we have  $\det(M + ab^T) = \det(M) (1 + b^T M^{-1} a)$ ; and  $\Delta = \Delta_1 + \Delta_2$  with  $\Delta_1 = w_2^T (\bar{U}^T U)^{-1} \bar{U}^T r$  and  $\Delta_2 = \|\tilde{r}\|^2 - \|r\|^2$ . Hence

$$\frac{\zeta_{t+1}}{\zeta_t} = \left( \frac{\det(\bar{U}^T U_{t+1})}{\det(\bar{U}^T U)} \right)^2 \geq 1 + \frac{\|r\|^2}{\|p\|^2} + 2 \frac{\Delta}{\|p\|^2}$$

■

We need the following results for the proof of the remaining intermediate results in Section 4.

**Theorem 25** *Vershynin (2010)* Let  $M$  be an  $N \times n$  matrix with i.i.d Gaussian entries distributed as  $\mathcal{N}(0, 1)$ . Then for every  $t \geq 0$ , with probability at least  $1 - 2 \exp(-t^2/2)$  one has

$$\sqrt{N} - \sqrt{n} - t \leq \sigma_{\min}(M) \leq \sigma_{\max}(M) \leq \sqrt{N} + \sqrt{n} + t.$$

**Lemma 26** Let  $A \in \mathbb{R}^{m \times n}$  be a Gaussian random matrix with i.i.d entries distributed as  $\mathcal{N}(0, 1/n)$  entries, then

$$\mathbb{E} \|Av_{\perp}\|^2 = \frac{m}{n} \|v_{\perp}\|^2 \quad \text{and} \quad \mathbb{E} \|\mathcal{P}_{AU}(Av_{\perp})\|^2 = \frac{d}{n} \|v_{\perp}\|^2$$

and for any  $\beta_2 > 1$  we have

$$\mathbb{P} \left( \|\mathcal{P}_{AU}(Av_{\perp})\|^2 \leq \beta_2 \frac{d}{n} \|v_{\perp}\|^2 \right) \geq 1 - \exp[-(\beta_2 - 1)^2 d/2].$$

**Lemma 27** Let  $A$  be a random matrix with i.i.d Gaussian entries distributed as  $\mathcal{N}(0, 1/n)$ , and let  $z_1, z_2 \in \mathbb{R}^n$  such that  $z_1 \perp z_2$ , then  $Az_1$  and  $Az_2$  are independent of each other.

The proofs of Lemma 26 and Lemma 27 are provided at the end of this section. Now we are ready to prove our intermediate results.

**Proof of Lemma 12** **Proof** Note that  $\|\tilde{r}\|^2 = \|(\mathbb{I}_m - \mathcal{P}_{AU})Av_\perp\|^2 = \|Av_\perp\|^2 - \|\mathcal{P}_{AU}(Av_\perp)\|^2$ . Therefore, applying Lemma 26 yields the first result.

For the second statement, note that  $\|r\|^2 = \|A^T(I - \mathcal{P}_{AU})Av_\perp\|^2 \leq \|A^TAv_\perp\|^2$ . Let  $A^T = [a_1, a_2, \dots, a_m]$  with  $a_i$  denoting the  $i^{\text{th}}$  row of  $A$ , then

$$\begin{aligned} \mathbb{E}[A^TAA^T] &= \mathbb{E}\left[\sum_{i=1}^m a_i a_i^T \sum_{j=1}^m a_j a_j^T\right] = \sum_{i=1}^m \mathbb{E}\left[a_i a_i^T \sum_{j \neq i}^m a_j a_j^T\right] + \sum_{i=1}^m \mathbb{E}[a_i a_i^T a_i a_i^T] \\ &\stackrel{\vartheta_1}{=} \left(\frac{m(m-1)}{n^2} + \frac{m}{n} \frac{n-1}{n} + \frac{3m}{n^2}\right) \mathbb{I}_n \\ &= \left(1 + \frac{m+1}{n}\right) \frac{m}{n} \mathbb{I}_n \end{aligned}$$

where  $\vartheta_1$  follows by

$$\sum_{i=1}^m \mathbb{E}\left[a_i a_i^T \sum_{j \neq i}^m a_j a_j^T\right] = \sum_{i=1}^m \mathbb{E}[a_i a_i^T] \mathbb{E}\left[\sum_{j \neq i}^m a_j a_j^T\right] = \frac{m(m-1)}{n^2}$$

and let  $g = a_i, i = 1, \dots, m$  and  $H = gg^T gg^T$ , then

$$\begin{aligned} \mathbb{E}[H_{kk}] &= \mathbb{E}[\|g\|^2 g_k^2] = \mathbb{E}[g_k^4] + \mathbb{E}[g_k^2] \mathbb{E}\sum_{j \neq k} g_j^2 = \frac{3}{n^2} + \frac{n-1}{n^2} \\ \mathbb{E}[H_{kl}] &= \mathbb{E}[\|g\|^2 g_k g_l] = \mathbb{E}[g_k g_l] \mathbb{E}\sum_{j \neq k, l} g_j^2 + \mathbb{E}[g_k^3 + g_l^3] = 0 \end{aligned}$$

Therefore,

$$\mathbb{E}[\|r\|^2 | v_\perp] \leq \mathbb{E}[v_\perp^T A^T A A^T A v_\perp | v_\perp] = \left(1 + \frac{m+1}{n}\right) \frac{m}{n} \|v_\perp\|^2$$

■

**Proof of Lemma 11** **Proof** Recall that  $\tilde{v}_\parallel = U w_2$ . According to Definition 23, we have  $\|\mathcal{P}_{AU}(Av_\perp)\|^2 = \|AU w_2\|^2$ . It then follows that

$$\|\mathcal{P}_{AU}(Av_\perp)\|^2 \geq \sigma_{\min}(AU)^2 \|\tilde{v}_\parallel\|^2 \quad (12)$$

For which  $AU \in \mathbb{R}^{m \times d}$  is a Gaussian random matrix with entries distributed as  $(AU)_{ij} \sim \mathcal{N}(0, 1/n)$ . Therefore if  $m \geq C_1 d \log n$ , Theorem 25 suggests that for any  $0 < \delta_1 < 1 - \sqrt{d/m}$  we have

$$\begin{aligned} \mathbb{P}\left(\sigma_{\min}(AU) \geq (1 - \delta_1) \sqrt{\frac{m}{n}} - \sqrt{\frac{d}{n}}\right) &\geq 1 - \exp\left\{-\frac{\delta_1^2 m}{2}\right\} \\ &\geq 1 - n^{-C_1 \delta_1^2 d/2} \end{aligned} \quad (13)$$

Setting  $\beta_2 = 1 + \delta_2 \sqrt{\frac{m}{d}}$  with  $\delta_2 > 0$  for Lemma 26, we have with probability at least  $1 - n^{-C_1 \delta_1^2 d/2} - n^{-C_1 \delta_2^2 d/2}$ ,

$$\|\tilde{v}_\parallel\|^2 \leq \frac{\|\mathcal{P}_{AU}(Av_\perp)\|^2}{\sigma_{\min}^2(AU)} \leq \frac{1 + \delta_2 \sqrt{m/d}}{(1 - \delta_1 - \sqrt{d/m})^2} \frac{d}{m} \|v_\perp\|^2$$

Let  $\eta_2 = \frac{\sqrt{d/m + \delta_2 \sqrt{d/m}}}{1 - \delta_1 - \sqrt{d/m}}$ , it then follows that

$$\|p\|^2 = \|v_\parallel + \tilde{v}_\parallel\|^2 \leq (\|v_\parallel\| + \|\tilde{v}_\parallel\|)^2 \leq (1 + \eta_2)^2 \|v\|^2$$

■

**Proof of Lemma 10** **Proof** Let  $U_\perp = \left[ \frac{v_\perp}{\|v_\perp\|}, C_\perp \right] \in \mathbb{R}^{n \times n-d}$  with orthonormal columns spanning the null space of  $U$ . Let  $M_1 = (\bar{U}^T U)^{-1} \bar{U}^T U_\perp$ , then

$$\begin{aligned} \mathbb{E} [\Delta_2 | v_\perp, U] &\stackrel{\vartheta_1}{=} \mathbb{E} \left[ w_2^T M_1 U_\perp^T A^T \tilde{r} \middle| v_\perp, U \right] \\ &= \text{tr} \left\{ M_1 \mathbb{E} \left( \left[ \frac{v_\perp}{\|v_\perp\|}, C_\perp \right]^T A^T \tilde{r} w_2^T \middle| v_\perp, U \right) \right\} \\ &\stackrel{\vartheta_2}{=} 0 \end{aligned} \tag{14}$$

where  $\vartheta_1$  holds since  $U \perp A^T \tilde{r}$ . To show  $\vartheta_2$ , let  $H_1 = v_\perp^T A^T \tilde{r} w_2^T$  and  $H_2 = C_\perp^T A^T \tilde{r} w_2^T$ . Note that according to Lemma 27,  $AU, Av_\perp, AC_\perp$  are independent of each other, therefore

$$\mathbb{E} [H_2 | v_\perp, U] = \mathbb{E} [C_\perp^T A^T] \mathbb{E} [\tilde{r} w_2^T | v_\perp, U] = 0 \tag{15}$$

and letting  $y = Av_\perp$

$$\begin{aligned} \mathbb{E} [H_1 | v_\perp, U] &= \frac{1}{\|v_\perp\|} \mathbb{E} \left[ y^T (\mathbb{I}_m - \mathcal{P}_{AU}) y y^T AU (U^T A^T AU)^{-1} \middle| U, v_\perp \right] \\ &= \frac{1}{\|v_\perp\|} \mathbb{E} \left[ \|(\mathbb{I}_m - \mathcal{P}_{AU}) y\|^2 y^T \middle| v_\perp, U \right] \mathbb{E} \left[ AU (U^T A^T AU)^{-1} \middle| U, v_\perp \right] \\ &\stackrel{\vartheta_3}{=} 0 \end{aligned} \tag{16}$$

where  $\vartheta_3$  follows because  $AU$  and  $y$  are independent, hence  $\mathcal{P}_{AU}y$  is the projection of the Gaussian random vector  $y$  onto a  $d$ -dimensional subspace that is independent of it. Since the Gaussian distribution is invariant under orthogonal rotation, this is equivalent to projecting  $y$  onto its first  $d$  coordinates. Together with the fact that the entries of  $y$  are i.i.d Gaussian random variables with mean zero, we have  $\mathbb{E} [\|(\mathbb{I}_m - \mathcal{P}_{AU}) y\|^2 y^T | v_\perp, U] = \mathbb{E} [\sum_{i=1}^d y_i^2 y^T | v_\perp, U] = 0$ .

(15) and (16) together yield (14), thus complete the proof. ■

**Proof of Lemma 26** **Proof**  $Av_\perp$  is a Gaussian random vector with i.i.d entries distributed as  $(Av_\perp)_i \sim \mathcal{N}(0, \|v_\perp\|^2/n)$ . Therefore,

$$\mathbb{E} \left[ \|Av_\perp\|^2 \mid v_\perp \right] = \frac{m}{n} \|v_\perp\|^2$$

According to Lemma 27 we have  $AU$  and  $Av_\perp$  are independent of each other, so are  $\mathcal{P}_{AU}$  and  $Av_\perp$ . Since Gaussian distribution is invariant under orthogonal rotation,  $\|\mathcal{P}_{AU}(Av_\perp)\|^2$  is equivalent to that of projecting  $Av_\perp$  onto its first  $d$  coordinates, i.e.,  $\|\mathcal{P}_{AU}(Av_\perp)\|^2 = \sum_{i=1}^d (Av_\perp)_i^2$ . Note that  $\frac{\sum_{i=1}^d (Av_\perp)_i^2}{\frac{d}{n} \|v_\perp\|^2}$  is a chi-squared distribution with degree  $d$ . It hence follows that,

$$\mathbb{E} \left[ \|\mathcal{P}_{AU}(Av_\perp)\|^2 \mid v_\perp \right] = \frac{d}{n} \|v_\perp\|^2$$

and for any  $\beta_2 > 1$  we have

$$\mathbb{P} \left( \|\mathcal{P}_{AU}(Av_\perp)\|^2 \geq \beta_2 \frac{d}{n} \|v_\perp\|^2 \right)$$

holds with probability at least  $1 - \exp(-(\beta_2 - 1)^2 d/2)$ . ■

**Proof of Lemma 27** **Proof** Let  $a_i$  denote the  $i^{th}$  column of  $A$ , and let  $H = Az_1 z_2^T A^T$ , then

$$\mathbb{E}[H_{ii}] = \mathbb{E}[a_i^T z_1 z_2^T a_i] = z_1^T \mathbb{E}[a_i a_i^T] z_2 = \frac{1}{n} z_1^T z_2 = 0$$

and

$$\mathbb{E}[H_{ij}] = \mathbb{E}[a_i^T z_1 z_2^T a_j] = z_1^T \mathbb{E}[a_i a_j^T] z_2 = 0$$

Therefore,  $\mathbb{E}[Az_1 z_2^T A^T] = \mathbb{E}[Az_1] \mathbb{E}[z_2^T A^T] = 0$ . Hence  $Az_1$  and  $Az_2$  are uncorrelated. Since  $Az_1$  and  $Az_2$  are Gaussian random vectors, they are independent of each other. ■

## Appendix D. Proof of Missing Data

**Proof of Lemma 15** **Proof** Note that for the missing data case we have  $\|\tilde{r}\|^2 = \|r_\Omega\|^2 = \|r\|^2$ . Therefore, following the same argument as that for Lemma 9 completes the proof. ■

We need the following lemmas for the proofs of the remaining intermediate results in Section 5, the proofs can be found in Balzano et al. (2010b).

**Lemma 28** *Balzano et al. (2010b)* For each iteration of GROUSE we have

$$\mathbb{E} \left[ \|v_{\perp, \Omega}\|^2 \mid v_\perp \right] = \frac{m}{n} \|v_\perp\|^2$$



**Lemma 29** *Balzano et al. (2010b)* Let  $\mu(U), \mu(v_\perp)$  denote the incoherence parameters of  $R(U)$  and  $v_\perp$ , and let  $\delta \in (0, 1)$  and  $\beta = \sqrt{2\mu(v_\perp) \log(1/\delta)}$ , then

$$\begin{aligned} \mathbb{E} \left[ \|U_\Omega^T v_{\perp, \Omega}\|_2^2 | v_\perp \right] &\leq \frac{m d \mu(U)}{n^2} \|v_\perp\|^2 \\ \mathbb{P} \left( \|U_\Omega^T v_{\perp, \Omega}\|_2^2 \leq (\beta + 1)^2 \frac{m d \mu(U)}{n^2} \|v_\perp\|^2 \right) &\geq 1 - \delta \end{aligned}$$

**Lemma 30** *Balzano et al. (2010b)* Suppose  $m \geq \frac{8}{3} d \mu(U) \log(2d/\delta)$ , then

$$\mathbb{P} \left( \left\| (U_\Omega^T U_\Omega)^{-1} \right\|_2 \leq \frac{n}{(1 - \gamma') m} \right) \geq 1 - \delta$$

where  $\gamma' = \sqrt{\frac{8 d \mu(U)}{3m} \log(2d/\delta)}$ .

Now we are ready for the proof of Lemma 16 and Lemma 17.

**Proof of Lemma 16** **Proof** Note that

$$\begin{aligned} \|r_\Omega\|^2 &= \|v_{\perp, \Omega}\|^2 - v_{\perp, \Omega}^T U_\Omega (U_\Omega^T U_\Omega)^{-1} U_\Omega^T v_{\perp, \Omega} \\ &\geq \|v_{\perp, \Omega}\|^2 - \left\| (U_\Omega^T U_\Omega)^{-1} \right\| \|\bar{U}_\Omega^T v_{\perp, \Omega}\|^2 \end{aligned}$$

Therefore, with probability at least  $1 - \delta$  we have

$$\begin{aligned} \mathbb{E} [\|r_\Omega\|^2 | v_\perp] &\geq \mathbb{E} [\|v_{\perp, \Omega}\|^2 | v_\perp] - \frac{n}{(1 - \gamma') m} \mathbb{E} [\|\bar{U}_\Omega^T v_{\perp, \Omega}\|^2 | v_\perp] \\ &\geq \frac{m}{n} \|v_\perp\|^2 - \frac{\mu(U)}{1 - \gamma'} \frac{d}{n} \|v_\perp\|^2 \end{aligned}$$

■

**Proof of Lemma 17** **Proof** Let  $\beta = \sqrt{2\mu(v_\perp) \log(1/\delta)}$ , then Lemma 29 and Lemma 30 together give the following

$$\begin{aligned} \|\tilde{v}_\parallel\|^2 &= \|U w_2\|^2 = \left\| (U_\Omega^T U_\Omega)^{-1} U_\Omega^T v_{\perp, \Omega} \right\|^2 \\ &\leq \left\| (U_\Omega^T U_\Omega)^{-1} \right\|^2 \|U_\Omega^T v_{\perp, \Omega}\|^2 \\ &\leq \frac{(\beta + 1)^2}{(1 - \gamma')^2} \frac{d \mu(U)}{m} \|v_\perp\|^2 \end{aligned}$$

holds with probability exceeding  $1 - 2\delta$ . It hence follows that

$$\|p\|^2 \leq (\|v_\parallel\| + \|\tilde{v}_\parallel\|)^2 \leq (1 + \eta'_3)^2 \|v\|^2$$

where  $\eta'_3 = \frac{\beta + 1}{1 - \gamma'} \sqrt{\frac{d \mu(U)}{m}}$ .

■

We also need the following lemma for the proof of Lemma 18, the proof of which is provided at the end of this section.

**Lemma 31** Let  $\gamma'_1 = \sqrt{\sin^2 \phi_d + \frac{d\mu_0}{nm}}$ , where again  $\mu_0$  denoting the incoherence parameter of  $R(\bar{U})$ . Then

$$\mathbb{E} [\|\bar{U}_\Omega^T v_{\perp, \Omega}\| | v_\perp] \leq \gamma'_1 \frac{m}{n} \|v_\perp\|$$

**Proof of Lemma 18** **Proof** Let  $\beta = \sqrt{2\mu(v_\perp) \log(1/\delta)}$ , then with probability at least  $1 - 2\delta$  we have

$$\begin{aligned} \mathbb{E} [w_2^T (\bar{U}^T U)^{-1} \bar{U}^T r | v_\perp] &= \mathbb{E} \left[ v_{\perp, \Omega}^T U_\Omega (U_\Omega^T U_\Omega)^{-1} (\bar{U}^T U)^{-1} \bar{U}_\Omega^T (I - \mathcal{P}_{U_\Omega}) v_{\perp, \Omega} | v_\perp \right] \\ &\leq \mathbb{E} \left[ \|v_{\perp, \Omega}^T U_\Omega\| \left\| (U_\Omega^T U_\Omega)^{-1} \right\| \left\| (\bar{U}^T U)^{-1} \right\| \left\| \bar{U}_\Omega^T (I - \mathcal{P}_{U_\Omega}) v_{\perp, \Omega} \right\| | v_\perp \right] \\ &\stackrel{\vartheta_1}{\leq} \frac{(1 + \beta)}{(1 - \gamma') \cos \phi_d} \frac{\sqrt{dm\mu(U)}}{m} \|v_\perp\| \mathbb{E} [\|\bar{U}_\Omega^T v_{\perp, \Omega}\| | v_\perp] \\ &= \frac{(1 + \beta)\gamma'_1}{(1 - \gamma') \cos \phi_d} \sqrt{\frac{d\mu(U)}{m}} \frac{m}{n} \|v_\perp\|^2 \end{aligned}$$

where  $\vartheta_1$  follows by Lemma Lemma 29 and 30, and the following  $\|\bar{U}_\Omega^T (I - \mathcal{P}_{U_\Omega}) v_{\perp, \Omega}\| \leq \|\bar{U}_\Omega^T v_{\perp, \Omega}\|$ ,  $\left\| (U_\Omega^T U_\Omega)^{-1} \right\| \leq \frac{1}{\cos \phi_{t,d}}$ .  $\blacksquare$

We also prove Lemma 19 for completeness. Before that we first call out the following lemma, the proof of which can be found in Balzano and Wright (2014).

**Lemma 32** Balzano and Wright (2014) There exists an orthogonal matrix  $V \in \mathbb{R}^{d \times d}$  such that

$$\sum_{k=1}^d \sin^2 \phi_k \leq \|\bar{U}V - U\|_F^2 \leq 2 \sum_{k=1}^d \sin^2 \phi_k$$

**Proof of Lemma 19** **Proof** According to Lemma 32 we have

$$\begin{aligned} \|U_i\|_2 &\leq \|\bar{U}_i\|_2 + \|\bar{U}_i V - U_i\|_2 \leq \|\bar{U}_i\| + \sqrt{2 \sum_{k=1}^d \sin^2 \phi_k} \\ &\leq \left(1 + \frac{1}{2\sqrt{2}}\right) \sqrt{\frac{d\mu_0}{n}} \end{aligned}$$

It hence follows that  $\|U_i\|_2^2 \leq 2\frac{d\mu_0}{n}$ .  $\blacksquare$

We use the following McDiarmid's inequality to prove Lemma 31.

**Theorem 33** (McDiarmid's Inequality McDiarmid (1989)). Let  $X_1, \dots, X_n$  be independent random variables, and assume  $f$  is a function for which there exist  $t_i, i = 1, \dots, n$  satisfying

$$\sup_{x_1, \dots, x_n, \hat{x}_i} |f(x_1, \dots, x_n) - f(x_1, \dots, \hat{x}_i, \dots, x_n)| \leq t_i$$

where  $\hat{x}_i$  indicates replacing the sample value  $x_i$  with any other of its possible values. Call  $f(X_1, \dots, X_n) := Y$ . Then for any  $\epsilon > 0$ ,

$$\begin{aligned}\mathbb{P}[Y \geq \mathbb{E}Y + \epsilon] &\leq \exp\left(-\frac{2\epsilon^2}{\sum_{i=1}^n t_i^2}\right) \\ \mathbb{P}[Y \leq \mathbb{E}Y - \epsilon] &\leq \exp\left(-\frac{2\epsilon^2}{\sum_{i=1}^n t_i^2}\right)\end{aligned}$$

**Proof of Lemma 31** **Proof** We use McDiarmid's inequality to prove this lemma. For the simplicity of notation denote  $v_\perp$  as  $y$ . Let  $X_i = \bar{U}_{\Omega(i)} y_{\Omega(i)}$ , and  $f(X_1, \dots, X_m) = \|\sum_{i=1}^m X_i\|_2 = \|\bar{U}_\Omega^T v_{\perp, \Omega}\|_2$ . Note that  $\|\bar{U}_{\Omega(i)}\|_2 = \|\mathcal{P}_{R(\bar{U})}(e_i)\| \leq \sqrt{d\mu_0/n}$ . Thus

$$\begin{aligned}\left\|\sum_{i=1}^m X_i\right\|_2 - \left\|\sum_{i \neq k}^m X_i + \hat{X}_k\right\|_2 &\leq \|X_k - \hat{X}_k\|_2 \leq \|X_k\|_2 + \|\hat{X}_k\|_2 \\ &\leq 2\|y\|_\infty \sqrt{d\mu_0/n}\end{aligned}\tag{17}$$

We next calculate  $\mathbb{E}[f(X_1, \dots, X_m)] = \mathbb{E}[\|\sum_{i=1}^m X_i\|_2]$ , for which we have

$$\begin{aligned}\mathbb{E}\|\bar{U}_\Omega^T y_\Omega\|^2 &= \mathbb{E}\left\langle \sum_{k=1}^m \sum_{i=1}^n \bar{U}_i y_i \mathbb{I}_{\{\Omega(k)=i\}}, \sum_{k=1}^m \sum_{i=1}^n \bar{U}_i y_i \mathbb{I}_{\{\Omega(k)=i\}} \right\rangle \\ &\stackrel{\vartheta_1}{=} \left\| \sum_{k=1}^m \sum_{i=1}^n \bar{U}_i y_i \frac{1}{n} \right\|^2 - \frac{1}{n^2} \sum_{k=1}^m \sum_{i=1}^n \left\langle \bar{U}_i y_i, \sum_{j \neq i} \bar{U}_j y_j \right\rangle \\ &= \frac{m^2}{n^2} \|\bar{U}^T y\|^2 - \frac{1}{n^2} \sum_{k=1}^m \sum_{i=1}^n \langle \bar{U}_i y_i, \bar{U}^T y \rangle + \frac{m}{n^2} \sum_{i=1}^n \|\bar{U}_i y_i\|^2 \\ &\leq \frac{m^2 - m}{n^2} \|\bar{U}^T y\|^2 + \frac{m}{n^2} \frac{d\mu_0}{n} \|y\|^2 \\ &\stackrel{\vartheta_2}{\leq} \frac{m^2}{n^2} \left( \sin^2 \phi_d + \frac{d\mu_0}{nm} \right) \|y\|^2\end{aligned}\tag{18}$$

where  $\vartheta_1$  follows by  $\mathbb{P}(\{\Omega(k) = i\} \cap \{\Omega(k) = j\}) = 0, \forall i \neq j$ ; and  $\vartheta_2$  holds since

$$\begin{aligned}\|\bar{U}^T y\|^2 &= \|\bar{U}^T (\mathbb{I} - UU^T) \bar{U} s\|^2 = s^T \bar{Y} \Sigma^4 \bar{Y}^T s \\ &\stackrel{\vartheta_3}{\leq} \sin^2 \phi_d s^T \bar{Y} \Sigma^2 \bar{Y}^T s = \sin^2 \phi_d \|v_\perp\|^2\end{aligned}$$

where  $\vartheta_3$  holds since  $\|v_\perp\|^2 = \|s\|^2 - v^T U U^T v = s^T \bar{Y} \Sigma^2 \bar{Y}^T s$ . ■

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